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DOMAINS OF ATTRACTION IN THE PROBLEM OF THE PLANE MOTION OF SYSTEMS WITH ROLLING*

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A class of mechanical systems with rolling, of gyrostat type, consisting of a main body (a carrier, platform or a vehicle frame) and bodies rolling on a fixed horizontal plane and imparting a longitudinal velocity $v = \text{const}$ to the main body, is considered. A non-linear mechanism of elastic interaction between the bodies in contact based on the Rocard /1/ axioms is adopted, and the side reactions are, in this case, monotonic functions of the drift angles. Qualitative analysis of the phase trajectories is used to show that the region of attraction of the zero solution of the equations of perturbed motion is unbounded. The conditions are found under which the domain of attraction is represented by the phase plane. A necessary condition for the domain of attraction of the unperturbed solution of a coarse dynamic system to be bounded is given in terms of the Poincaré index for the singularities lying at the boundary of the domain of attraction. A characteristic phase pattern is constructed for specific values of the system parameters.

The problem of plane parallel motion of a constrained gyrostat with rolling was formulated in /2/. We shall therefore limit ourselves to a brief explanation concerning the purpose of the present study. Two geometrically and dynamically symmetrical bodies rotate about two parallel axes rigidly coupled to the main body. The rotating bodies are in contact with a fixed horizontal plane. We denote by x and y the abscissa and ordinate of the centre of mass D of the gyrostat in the inertial coordinate system, ϑ is the course angle, $\omega = \vartheta^{\cdot}$, $v = x^{\cdot} \cos \vartheta + y^{\cdot} \sin \vartheta$, $u = -x^{\cdot} \sin \vartheta + y^{\cdot} \cos \vartheta$ are the longitudinal and transverse velocities of the point D (the quantities v and u are quasivelocities), m and I are the mass and central vertical moment of inertia of the gyrostat, l_1, l_2 are the distances between the point D and the middle of the front and back axis respectively, and $l = l_1 + l_2$, N_1, N_2 are the static components of the vertical reactions. If the bodies carried are inertialess and elastically deformable, they serve as a source of specific forces acting on the main body. According to Rocard /1/ the side reactions Y_i are functions of the so-called drift angles

$$\delta_1 = -\operatorname{arctg} [(u + l_1\omega)/v],\ \delta_2 = \operatorname{arctg} [(-u + l_2\omega)/v]$$

We shall restrict ourselves to the case of monotonic dependence $Y_i = Y_i(\delta_i)$, assuming that

$$k_i(\delta_i) \equiv d(Y_i/N_i)/d\delta_i > 0, \quad \lim_{\delta_i \to +\infty} Y_i/N_i = c_i = \text{const} \tag{1}$$
$$Y_i(0) = 0, \quad Y_i'(0) = a_i$$

and write the equations of motion (1) of /2/ in the form

$$\omega^{\cdot} = P(\omega, u),\ u^{\cdot} = Q(\omega, u) \tag{2}$$
$$P(\omega, u) = (Y_1 l_1 - Y_2 l_2)/I,\ Q(\omega, u) = (Y_1 + Y_2)/m - v\omega$$

System (2) describes the motion of the representative point (ω, u) in the two-dimensional phase space $F(\omega, u)$. The equilibrium state in the space F corresponds to the steady motion of the gyrostat. The number of such states depends on the form of the function $Y_i(\delta_i)$ /2/. We take the rectilinear motion along the Ox axis as the unperturbed motion. The point (0,0) corresponds to the unperturbed solution of (2) in the ωu plane. Equations (2) for this point will represent the equations in variations. In the three-dimensional phase space $\Phi(U, \vartheta, \vartheta^{\cdot})$ where $U = y^{\cdot}$, the zero solution of (2) corresponds to the one-dimensional manifold $U = v\vartheta$, $\vartheta^{\cdot} = 0$ of equilibrium states /3/. The physical meaning of the latter consists of the arbitrariness of the straight line along which a steady motion of the gyrostat is possible.

We know /1/ that when $a_1 l_1 - a_2 l_2 < 0$ (insufficient rotatability), i.e. when $k_1(0) < k_2(0)$ /2/, the origin of coordinates of the ωu plane is asymptotically stable for $\forall v > 0$, while when $a_1 l_1 - a_2 l_2 > 0$ (excessive rotatability) it is stable only for $v < v_+ = l\,[a_1 a_2 m^{-1}(a_1 l_1 - a_2 l_2)^{-1}]^{1/2}$. A proposition is developed in /2/ stating that the loss of stability of the origin of coordinates regarded as a singularity of the equations of perturbed motion of a rolling gyrostat with

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excessive rotatability is connected, at the supercritical velocities, with the behaviour of the singularities different from the origin of coordinates. In the present paper the problem of the structure of the domains of attraction of the zero solution is solved by analysing the phase trajectories of system (2) with conditions (1). Just as the conditions of asymptotic Lyapunov stability depend on the relation connecting k_1 and k_2, which determine the behaviour of the functions $Y_i(\delta_i)$ at the zero, so the form of the domains of attraction depends on the relations connecting c_1 with c_2 determining the behaviour of the functions $Y_i(\delta_i)$ at infinity. The construction of the latter is based on the following assertion.

Theorem. If the origin of coordinates is asymptotically stable and conditions (1) are satisfied, then the domain of attraction of the unperturbed solution is unbounded. The whole phase plane will be a domain of attraction if and only if $c_1 < c_2$.

Proof. Let us find the expression for the divergence of the vector velocity field defined by system (2)

$$\operatorname{div}(P, Q) = -\frac{1}{v}\left\{\left(\frac{l_1^2}{I} + \frac{1}{m}\right)\frac{1}{1+[(u+l_1\omega)/v]^2}\frac{dY_1}{d\delta_1} + \left(\frac{l_2^2}{I} + \frac{1}{m}\right)\frac{1}{1+[(-u+l_2\omega/v]^2}\frac{dY_2}{d\delta_2}\right\}$$

By virtue of the first condition of (1) we have $\operatorname{div}(P, Q) < 0$ on the whole phase plane. According to the Bendickson criterion this implies that there are no closed contours wholly composed of the phase trajectories of system (2) in any singly connected region of the phase plane. In /4/ it was shown that the boundary of the whole domain of attraction of the unperturbed solution of the autonomous system consists of whole trajectories, or of segments of trajectories. Consequently the domain of attraction of the zero solution of (2) is unbounded.

To prove the second part of the theorem we shall analyse the behaviour of the phase trajectories in the ωu plane. Let us first determine the sets of points at which $\omega^{\cdot} = 0$ or $u^{\cdot} = 0$.

The condition $\omega^{\cdot} = 0$ reduces to the relation $Y_1/N_1 = Y_2/N_2$ where $N_1 = mgl_2/l$, $N_2 = mgl_1/l$. Since the geometrical position of the points in the ωu plane corresponding to the condition $\operatorname{sign}\delta_1 = \operatorname{sign}\delta_2$ lies between the straight lines $u = -l_1\omega$, $u = l_2\omega$, and the relation $\delta_1 = \delta_2$ is true at every point of the u axis, it follows that the set $\omega^{\cdot} = 0$ in question lies totally between the straight lines $u = -l_1\omega$, $u = l_2\omega$ and passes through the origin of coordinates. When $u_0 = -v \operatorname{tg}\delta_0$, the set intersects the u axis, provided that the graphs depicting the depedence $Y_i(\delta_i)/N_i$ have a point of intersection when $\delta_1 = \delta_2 = \delta_0 \neq 0$.

The curve

$$l_1Y_1\left(-\operatorname{arctg}\frac{u+l_1\omega}{v}\right) - l_2Y_2\left(\operatorname{arctg}\frac{-u+l_2\omega}{v}\right) = 0 \tag{3}$$

has the following angular coefficient at the origin of coordinates:

$$k_\omega = -(a_1l_1^2 + a_2l_2^2)/(a_1l_1 - a_2l_2)$$

When $k_1(0) < k_2(0)$, we have $k_\omega > 0$ and if the graphs depiciting the dependence $Y_i(\delta_i)/N_i$ have no points of intersection, then the whole curve $\{(\omega, u) : \omega^{\cdot} = 0\}$ lies in the first and third quadrant. When $k_1(0) > k_2(0)$ and there are no intersections, curve (3) lies in the second and fourth quadrant.

We have the following relation for the set of points of the phase plane at which $u^{\cdot} = 0$:

$$Y_1\left(-\operatorname{arctg}\frac{u+l_1\omega}{v}\right) + Y_2\left(\operatorname{arctg}\frac{-u+l_2\omega}{v}\right) = mv\omega \tag{4}$$

and from this it follows that the curve $\{(\omega, u): u^{\cdot} = 0\}$ tends to the vertical asymptote $\omega = \omega_*$ and $\omega = -\omega_*$ as $u \to -\infty$ and $u \to +\infty$ respectively. Here $\omega_* = (c_1N_1 + c_2N_2)\, m^{-1}v^{-1}$. The angular coefficient of the curve is given at the origin of coordinates by the expression

$$k_u = -(mv^2 + a_1l_1 - a_2l_2)/(a_1 + a_2)$$

If $a_1l_1 - a_2l_2 < 0$ and $v < v_1$, then curve (4) intersects the ω axis when $\omega = \pm\omega_1$. Here we have

$$v_1^2 = (a_2l_2 - a_1l_1)\, m^{-1},\quad mv\omega_1 = Y_1(-\operatorname{arctg} l_1v^{-1}\omega_1) + Y_2(\operatorname{arctg} l_2v^{-1}\omega_1)$$

If on the other hand $a_1l_1 - a_2l_2 < 0$ and $v > v_1$, or $a_1l_1 - a_2l_2 > 0$, then curve (4) lies wholly in the second and fourth quadrant.

The points of intersection of (3) and (4) represent the singularities of the system (2). The relative position of these curves near the singularity determines their type. Thus we have the following results for the origin of coordinates. Let us denote by α_ω, α_u the angle of inclination of the curves (3), and hence of (4), to the abscissa at the origin of coordinates.

Let $a_1l_1 - a_2l_2 > 0$, whereupon $k_u < 0$, $k_\omega < 0$. If $\pi/2 < \alpha_\omega < \alpha_u < \pi$, then the direction of the phase trajectories of system (2) in a sufficiently small neighbourhood of the origin of coordinates is such, that the point (0,0) is a stable node. If on the other hand $\pi/2 < \alpha_u < \alpha_\omega < \pi$, then the origin of coordinates is a saddle. The change from the node to the saddle takes place when $v = v_+$.

When $a_1l_1 - a_2l_2 < 0$ we have $k_\omega > 0$, $k_u = k_u(v)$ changes its sign on passing through $v = v_1$, and $k_u(0) > k_\omega$. When $v < v_1$ we have $0 < \alpha_u < \alpha_\omega < \pi/2$, and the point (0,0) is a stable node. When $v \in (v_1, v_*)$ where

$$v_*^2 = I\{[(a_1l_1^2 + a_2l_2^2)/I + (a_1 + a_2)/m]^2/4 - a_1a_2l^2m^{-1}I^{-1}\}(a_2l_2 - a_1l_1)^{-1}$$

we have $0 < \alpha_\omega < \pi/2$, $\pi/2 < \alpha_u < \pi$. In this case an isocline k_i exists, situated below the isocline k_ω which "knocks" the phase trajectories into the origin of coordinates. Otherwise the node at the origin of coordinates would be transformed at $v = v_1$ into a focus, and this is impossible since the linear theory tells us that the point (0,0) becomes a focus when $v > v_* > v_1$. If $v > v_*$, then the direction of the phase trajectories of system (2) on the curves (3) and (4) are the same as in the case $v_1 < v < v_*$, but the origin of coordinates becomes a stable focus.

When $a_1l_1 - a_2l_2 = 0$ we always have $k_u < 0$, $v_1 = 0$, $k_\omega = \infty$ and the point (0,0) is a stable node when $0 < v < v_*$, and a stable focus when $v > v_*$.

Let us now analyse the possible relative dispositions of the curves (3) and (4) over the whole ωu plane.

1°. If $k_1(0) > k_2(0)$, $c_1 > c_2$, i.e. the dependence of the specific side reactions Y_i/N_i on the drift angles δ_i has the form shown in Fig.1 (here and henceforth a dashed line corresponds to $i = 1$ and the solid line to $i = 2$), then from the behaviour of the curve $Y = Y(\delta_2 - \delta_1)$ it follows /2/ that when $v < v_+$, we have a pair of singularities (in the second and fourth quadrant) in the ωu plane. The singularities represent the points of intersection of the curves (3) (the solid line in the lower part of Fig.1) and (4) (the dashed line). From the direction

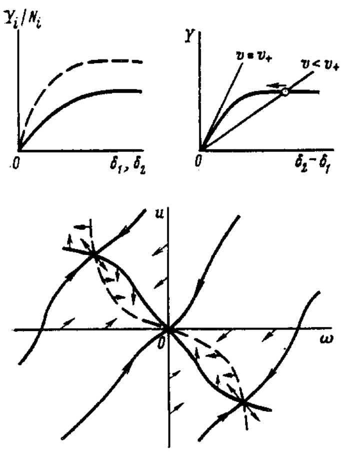

Fig.1

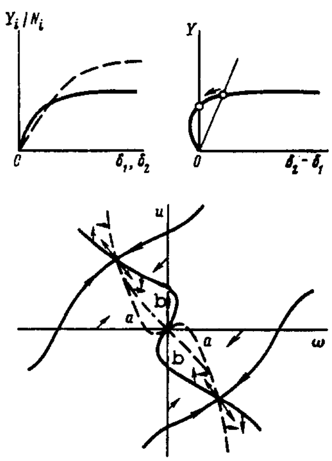

Fig.2

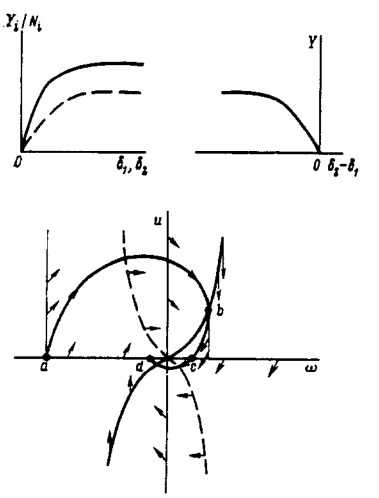

Fig.3

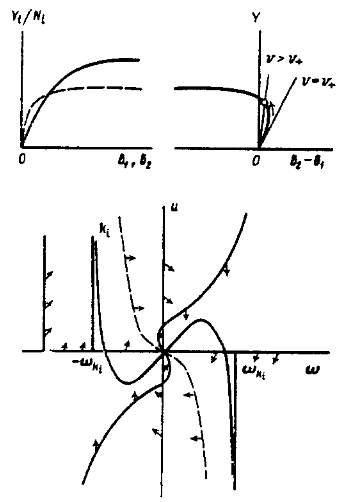

Fig.4

of the vector field it follows that the singuarities are saddles /5/ and also, that phase trajectories exist which lie wholly in the first and third quadrant and go to the origin of coordinates. When the velocity v increases, the saddle points move towards the origin of coordinates and merge with it when $v = v_+$. When $v > v_+$, we have a unique saddle point at the origin of coordinates. The boundary of the domain of attraction of the zero solution of (2) represents, at $v < v_+$, the separatrices entering the saddle points. The separatrices define, within the second and fourth qudrant, bounded regions such that the phase trajectories lying within them must remain there. The separatrices emerging from the saddle points and lying within the regions indicated have the origin of coordinates as the ω-limit point /6/.

2°. When $k_1(0) < k_2(0), c_1 > c_2$, the dependence of the quantities Y_i/N_i on δ_i is shown in Fig.2. In this case we also have a pair of saddle points, but when v increases, the saddle points cannot arrive at the origin of coordinates to upset its stability. Their limiting position as $v \to +\infty$ corresponds to the point of intersection of the curve (3) with the u axis. The separatrices arriving at the saddle points restirct the domain of attraction of the origin of coordinates. When the velocity v increases, (4) changes its form from a to b, losing its extrema.

3°. When $k_1(0) < k_2(0), c_1 < c_2$ (Fig.3) a unique singularity exists at the origin of coordinates, asymptotically stable when $0 < v < +\infty$. The point (0,0) can be either a node, or a focus. Let us consider the case of a focus. Using the properties of the vector velocity field we can show that the domain of attraction of the origin of coordinates consists of the whole phase plane. Indeed, the phase trajectory having emerged from the position a arrives at point b on (3), then at point c on the ω axis, and by virtue of the symmetry of the properties of the vector field, at point d.

We shall use the method of reductio ad absurdum to show that the spiral coils in. If this were not true, if for some initial values the spiral ad had uncoiled $(d < a < 0)$, then the spiral and the segment $[a, d]$ together would form a closed contour on the ω axis which would not contain even a single trajectory. This contradicts the fact that the domain of attraction of the origin of coordinates is unbounded.

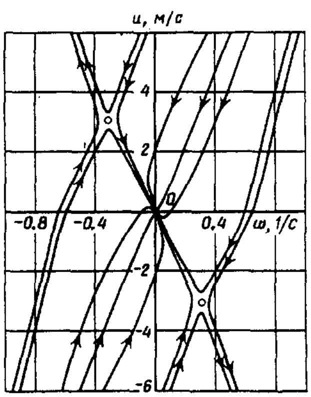

Fig.5

A situation in which the origin of coordinates is a node, is analogous to the one described above for subcritical velocities.

4°. Let $k_1(0) > k_2(0), c_1 < c_2$ (Fig.4). When $v < v_+$, we have a unique, asymptotically stable singularity of (2), namely a node at the origin of coordinates. The phase trajectories at infinity coil into spirals. This follows from the fact that the isocline k_i, i.e. the curve $Q(\omega, u)/P(\omega, u) = \text{const} > 0$, has the form shown in the lower part of Fig.4. The fact that the isoclines have vertical asymptotes when ω tends to some finite value ω_{k_i}, follows from the fact that the forces and moments in (2) are bounded functions of the coordinates. If we assume that the spirals in question uncoil, we shall arrive at the contradiction mentioned in 3°. Thus the domain of attraction constitutes the whole phase plane, This proves the theorem.

Fig.5 shows the behaviour of the actual phase trajectories of system (2), The figure is constructed for $a_1 = a_2 = 57300$H, $m = 2527$ kg, $I = 6550$ kgm^{-2}, $l_1 = 1.73$ m, $l_2 = 1.5$ m, $l = 3.23$ m, $c_1 = 0.8$, $c_2 = 0.7$, $v = 19$ m/sec.

In conclusion we note that the requirement that the divergence of the vector field should be negative over the whole field is not necessary for the domain of attraction of the unperturbed solution of (2) to be unbounded. The necessary condition for the boundary of the domain of attraction of the unperturbed solution of the coarse system to be a closed contour is that the sum of the Poincaré indices of the singularities lying on the boundary of the domain of attraction should be zero. This follows from the fact that the boundary of the domain of attraction may contain only unstable singularities (just as in coarse dynamic systems the separatrix cannot pass from a saddle to a saddle), and the boundary of the domain of attraction contains only the whiskers entering the saddles. A one-to-one correspondence must exist between the singularities with index one, and the saddles.

We shall illustrate this by an example. It may occur in mechanical systems with rolling, that when $k_1(0) > k_2(0), c_1 > c_2, v < v_+$, the relations $Y_i(\delta_i)/N_i$ will have decreasing segments, although $c_1 > \max Y_2/N_2$. Then the character of the vector field of (2) will be the same as in case 1° (for the monotonically increasing functions $Y_i(\delta_i)/N_i$). Now, the divergence of the vector field is negative only in a finite neighbourhood of the origin of coordinates. By virtue of the character of the vector field (we have the phase trajectories moving from infinity to the origin of coordinates in the first and third quadrant) system (2) cannot have closed trajectories. The boundary of the domain of attraction cannot be a closed contour and it contains, just as in 1°, saddle points. In this case the sum of the indices of the singularities lying at the boundary is different from zero.

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THE MOTION OF A SYSTEM OF VORTEX RINGS IN AN INCOMPRESSIBLE FLUID*

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Hamiltonian formalism is developed in the problem of the motion of a system of coaxial vortex rings in an infinite, incompressible ideal fluid. An additional invariant of the motion representing the momentum of the surrounding fluid, is determined. In the case of two vortex rings the equations of motion are found to be completely integrable, and this explains the mutual slip-through of the vortex rings described qualitatively by Helmholtz. The influence of viscosity on the initial stage of motion is assessed.

1. Hamiltonian formulation. The problem of the motion of vortex rings which has already been studied in the last century, represents the simplest case of a three-dimensional vortex flow. Even in this simplest case a theoretical analysis is possible only when the radius of the vortex ring is much greater than the radius of the vortex core. Let us consider a system of coaxial vortex rings moving through an infinite, ideal incompressible fluid at rest at infinity. We shall introduce a cylindrical r, z, θ-coordinate system where the z axis is directed along the general axis of the vortex rings. Let Γ_α be the circulation of the vortex ring with index $\alpha, \alpha = 1, \ldots, N$, R_α be the ring radius, a_α the radius of the vortex core and z_α the longitudinal coordinate of the vortex. We shall seek the veclocity field outside the vortex rings in the form

$$\mathbf{v} = \operatorname{rot} \mathbf{A} \tag{1.1}$$

Symmetry considerations imply that $\mathbf{A} = A(r, z)\mathbf{e}_\theta$. Then from (1.1) we obtain

$$v_r = -\frac{\partial A}{\partial z}, \quad v_z = \frac{1}{r}\frac{\partial (rA)}{\partial r} \tag{1.2}$$

Substituting (1.2) into the equation

$$\operatorname{rot} \mathbf{v} = \sum_{\alpha=1}^{N} \Gamma_\alpha \delta(r - R_\alpha)\, \delta(z - z_\alpha)$$

we arrive at the following equation for the vector potential A:

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial z^2} - \frac{A}{r^2} = -\sum_{\alpha=1}^{N} \Gamma_\alpha \delta(r - R_\alpha)\, \delta(z - z_\alpha) \tag{1.3}$$

The right-hand side of (1.3) is obtained under the assumption that $a/R \ll 1$, and expresses the fact that the circulation along any closed contour enclosing the vortex with index α is equal to Γ_α. Since (1.3) is linear, it follows that the solution can be expressed as the sum of solutions for the isolated vortex rings, and has the form

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